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Oscillating Polynomials and Approximations to Fractional Powers of x

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1. INTRODUCTION

In this note we give a brief account of oscillating polynomials and consider the problem of approximations to $x^{1/k}$, where k is a positive integer. Let c_i be real for all i and write

$$E_{n,k} = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^{1/k} - (c_0 + c_1 x + \cdots + c_n x^n)|. \tag{1.1}$$

The properties of oscillating polynomials enable us to give a lower bound for $E_{n,k}$ for all $n \geqslant 2$ and $k \geqslant 3$. We use an iteration process (cf. [3]) to find numerical values of $E_{n,k}$ for some n and k.

2. OSCILLATING POLYNOMIALS

Let $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_n$ be a given set of integers. Then

$$p(x) = c_0 x^{\alpha_0} + c_1 x^{\alpha_1} + \cdots + c_n x^{\alpha_n}, \tag{2.1}$$

where c_i are all real and $\prod_{i=0}^{n} c_i \neq 0$, is said to be an oscillating polynomial (o.p.) in $[0,1]$ with exponents $\alpha_0, \ldots, \alpha_n$ if $\max_{[0,1]} |p(x)|$ is attained for $n+1$ values of x in $[0,1]$. The following properties of oscillating polynomials are necessary for our work. (See [1], [2].)

(i) To a given set of exponents, there corresponds an o.p. in $[0,1]$ which is unique except for a constant factor.

(ii) Write $M = \max_{[0,1]} |p(x)|$. An o.p. $p(x)$ assumes the values $\pm M$ alternately at $n+1$ points in $[0,1]$.

(iii) Let $p(x) = \sum_{j=0}^{n} A_j x^{\alpha_j}$ be an o.p. in $[0,1]$ and let $q(x) = \sum_{j=0}^{n} B_j x^{\alpha_j}$ (all B_j real) be another polynomial. Suppose $B_j = A_j$ for at least one j, where $\alpha_j > 0$. Then $\max_{[0,1]} |q(x)| > \max_{[0,1]} |p(x)|$.

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(iv) Let $p(x) = a_0 x^{\alpha_0} + \sum_{k=1}^{n} a_k x^{\alpha_k}$ and $q(x) = a_0 x^{\alpha_0} + \sum_{k=1}^{n} b_k x^{\beta_k}$ be oscillating polynomials such that $0 < \alpha_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n$. Then $\max_{[0,1]} |p(x)| < \max_{[0,1]} |q(x)|$.

3. Three Exponents

We now calculate an o.p. $p(x)$ with three exponents 0, h, k ($h < k$). Write $p(x) = a_0 + a_1 x^h + a_2 x^k$, $a_0 a_1 a_2 \neq 0$. Then $p'(x) = (ha_1 + ka_2 x^{k-h}) x^{h-1}$ and so $p'(x_1) = 0$, where $x_1 = (-a_1 h/ka_2)^{1/(k-h)}$. Since $p(0) = a_0$, $p(x_1) = -a_0$ and $p(1) = a_0 + a_1 + a_2 = a_0$, we have

$$p(x) = a_0 \left\{ 1 + 2\left(\frac{k}{h-k}\right)\left(\frac{k}{h}\right)^{h/(k-h)} (x^h - x^k) \right\},$$

and $M = |a_0|$.

4. Four Exponents

To derive an o.p. with exponents 0, h, $2h$, $4h$, we first consider

$$p(x) = 1 + b_1 x + b_2 x^2 + b_3 x^4, \qquad b_1 b_2 b_3 \neq 0. \tag{4.1}$$

Then $p(0) = 1$, $p(x_1) = -1$, $p(x_2) = 1$, $p(1) = -1$ where $0 < x_1 < x_2 < 1$ and $p'(x_1) = p'(x_2) = 0$. Eliminating b_1, b_2, b_3 and x_2, we obtain

$$54x_1^5 + 99x_1^4 + 64x_1^3 + 6x_1^2 - 6x_1 - 1 = 0. \tag{4.2}$$

Since $-\frac{1}{2}$, which lies outside $[0, 1]$, is a root of (4.2), we are able to write

$$x_1^4 + \tfrac{4}{3}x_1^3 + \tfrac{14}{27}x_1^2 - \tfrac{4}{27}x_1 - \tfrac{1}{27} = 0.$$

This equation has two nonreal roots, one negative root and one positive root. Hence

$$x_1 = \tfrac{1}{9}[2\sqrt{3} - 3 + \sqrt{6}(\sqrt{3} - 1)^{1/2}] = 0.28443158\ldots. \tag{4.3}$$

Furthermore

$$\begin{aligned} b_1 &= \frac{-4(1 + x_1)^2}{x_1(1 + 2x_1)} = -14.788327\ldots \\ b_3 &= \frac{-2}{x_1^2(2x_1 + 1)} = -15.757572\ldots \\ b_2 &= -(2 + b_1 + b_3) = 28.545899\ldots. \end{aligned} \tag{4.4}$$

Note that $1 + b_1 x^h + b_2 x^{2h} + b_3 x^{4h} = p_h(x)$ is an o.p. with exponents 0, h, $2h$ and $4h$. Finally $Mp_h(x)$ and $-Mp_h(x)$ are both oscillating polynomials with exponents 0, h, $2h$, $4h$ such that

$$\max_{[0,1]} |\pm Mp_h(x)| = M.$$

5. VALUES FOR $E_{n,k}$, $2 \leqslant k \leqslant 5$

(i) Suppose that

$$p^*_{n,k}(y) = a_0 + y + a_1 y^k + a_2 y^{2k} + \cdots + a_n y^{nk} \tag{5.1}$$

is the o.p. with exponents 0, 1, k, $2k$, ..., nk. Consider the case $k = 2$ and write $y^2 = x$. Then

$$E_{n,2} = \max_{[0,1]} \left| (x)^{1/2} - \left(-\sum_{i=0}^{n} a_i x^i \right) \right|. \tag{5.2}$$

Note that

$$E_{n,2} = \max_{[-1,1]} \left| |x| - \left(-\sum_{i=0}^{n} a_i x^{2i} \right) \right|$$

$$= \min_{c_i} \max_{[-1,1]} \left| |x| - \left(\sum_{i=0}^{2n} c_i x^i \right) \right|.$$

From (4.1) we have

$$p^*_{2,2}(y) = 1/b_1 + y + (b_2/b_1) y^2 + (b_3/b_1) y^4,$$

where b_1, b_2, b_3 are given by (4.4). Also

$$E_{2,2} = 1/|b_1| = .06762\ldots. \tag{5.3}$$

We shall write for the o.p. with exponents 0, 1, k, $2k$, ..., nk and constant term equal to 1,

$$p_{n,k}(y) = \frac{1}{a_0} p^*_{n,k}(y) = 1 + y/a_0 + (a_1/a_0) y^k + \cdots + (a_n/a_0) y^{nk}. \tag{5.4}$$

(ii) The values of $E_{n,2}$, when $3 \leqslant n \leqslant 23$, and the oscillating polynomials for $3 \leqslant n \leqslant 12$, are tabulated in [3]. We list below the oscillating polynomials $p_{n,k}$ and the values of $E_{n,k}$ when $k = 3, 4, 5$ and $n = 1, 2, 3$.

$p_{1,3}(x) = 1 - 5.19615x + 5.19615x^3$

$E_{1,3} = .19245$

$p_{2,3}(x) = 1 - 7.87889x + 18.8545x^3 - 12.9757x^6$

$E_{2,3} = .126921$

$p_{3,3}(x) = 1 - 10.2167x + 41.4544x^3 - 75.4902x^6 + 44.2526x^9$

$E_{3,3} = .097879$

$p_{1,4}(x) = 1 - 4.23307x + 4.23307x^4$

$E_{1,4} = .236235$

$p_{2,4}(x) = 1 - 5.80871x + 15.5433x^4 - 11.7346x^8$

$E_{2,4} = .172155$

$p_{3,4}(x) = 1 - 7.06647x + 34.2885x^4 - 68.6003x^8 + 41.3782x^{12}$

$E_{3,4} = .141513$

$p_{1,5}(x) = 1 - 3.73837x + 3.73837x^5$

$E_{1,5} = .267496$

$p_{2,5}(x) = 1 - 4.82798x + 13.8493x^6 - 11.0214x^{10}$

$E_{2,5} = .207126$

$p_{3,5}(x) = 1 - 5.65136x + 30.6244x^5 - 64.6469x^{10} + 39.6738x^{15}$

$E_{3,5} = .176949$

By direct calculations one can easily verify the following:

$$E_{1,3} = 3^{-3/2}, \qquad E_{1,4} = (3/2)4^{-4/3}, \qquad E_{1,5} = (2)5^{-5/4}.$$

6. Lower Bound for $E_{n,3}$, $n \geqslant 2$

Let $r(x)$ be the o.p. $x + c_2x^6 + c_3x^9 + \cdots + c_nx^{3n}$ with exponents $1, 6, 9, \ldots, 3n$ and let $q(x)$ be the o.p.

$$x + b_2x^5 + b_3x^7 + b_4x^{11} + \cdots + b_ix^{\alpha_i} + \cdots + b_nx^{\alpha_n},$$

where α_i is the odd integer between $3i-3$ and $3i$. By property (iv),

$$\max |r(x)| > \max |q(x)|.$$

Since $q(x)$ has fewer exponents than the Chebyshev polynomial $T_{\alpha_n}(x)$ which is the o.p. with exponents $1, 3, 5, \ldots, \alpha_n$, we have

$$\max_{[0,1]} |q(x)| > \max_{[0,1]} \left|\frac{T_{\alpha_n}(x)}{\alpha_n}\right| \geqslant \frac{1}{3n-1}.$$

Hence for any polynomial

$$S(x) = x + b_2x^6 + b_3x^9 + \cdots + b_nx^{3n}, \tag{6.1}$$

$$\max_{[0,1]} |S(x)| > 1/(3n-1). \tag{6.2}$$

Now write

$$E'_{n,3} = \min_{a_i} \max_{[0,1]} |x + a_1x^3 + a_2x^6 + \cdots + a_nx^{3n}|.$$

It is easily seen (cf. [2], p. 29) that

$$E_{n,3} > E'_{n,3}/2. \tag{6.3}$$

Consider the o.p. $p(x) = x + a_1 x^3 + a_2 x^6 + \cdots + a_n x^{3n}$. For each $x \in [0,1]$ and for $\mu > 0$, we have

$$\left|p\left(\frac{x}{1+\mu}\right)\right| = \left|\frac{x}{1+\mu} + a_1\left(\frac{x}{1+\mu}\right)^3 + a_2\left(\frac{x}{1+\mu}\right)^6 + \cdots + a_n\left(\frac{x}{1+\mu}\right)^{3n}\right| \leqslant E'_{n,3},$$

or

$$\left|x(1+\mu)^2 + a_1 x^3 + a_2 \frac{x^6}{(1+\mu)^3} + \cdots + a_n \frac{x^{3n}}{(1+\mu)^{3n-3}}\right| \leqslant (1+\mu)^3 E'_{n,3}.$$

This inequality can be written as

$$|p(x) + (2\mu + \mu^2)(x + b_2 x^6 + b_3 x^9 + \cdots + b_n x^{3n})| \leqslant (1+\mu)^3 E'_{n,3}.$$

Hence

$$-|p(x)| + (2\mu + \mu^2)|S(x)| \leqslant (1+\mu)^3 E'_{n,3}, \tag{6.4}$$

where $S(x)$ is a polynomial of the form given in (6.1). Since (6.4) holds for all $x \in [0,1]$,

$$(2\mu + \mu^2) \max_{[0,\,1]} |S(x)| \leqslant E'_{n,3}\{(1+\mu)^3 + 1\}. \tag{6.5}$$

From (6.2) and (6.5) we obtain

$$E'_{n,3} > 1/(3n-1)(2\mu + \mu^2)/((1+\mu)^3 + 1).$$

Taking $\mu = 1$, we get

$$E'_{n,3} > \frac{1}{3(3n-1)},$$

and hence, from (6.3) we have

$$E_{n,3} > 1/6(3n-1). \tag{6.6}$$

7. Lower Bound for $E_{n,k}$, $k \geqslant 4$

Consider the o.p. $q_{n,k}(x) = x + a_1 x^k + a_2 x^{2k} + \cdots + a_n x^{nk}$, $n \geqslant 2$, $k \geqslant 4$. Let $\alpha_1 = 3$ and, for $i = 2, 3, \ldots, n$, let

$$\alpha_i = \begin{cases} (i-1)k + 1 & \text{if } k \text{ is even or } i \text{ is odd,} \\ (i-1)k + 2 & \text{otherwise.} \end{cases}$$

Note that the α_i are all odd. Let $r(x)$ be the o.p.

$$r(x) = x + b_1 x^{\alpha_1} + b_2 x^{\alpha_2} + \cdots + b_n x^{\alpha_n}.$$

By property (iv),

$$\max_{[0,1]} |q_{n,k}(x)| > \max_{[0,1]} |r(x)|$$
$$> \max_{[0,1]} \left|\frac{T_{\alpha_n}(x)}{\alpha_n}\right| = \frac{1}{\alpha_n}.$$

As in (6.3), $E_{n,k} > E'_{n,k}/2$, and so we get, for $n \geqslant 2$, $k \geqslant 4$,

$$E_{n,k} > \begin{cases} \dfrac{1}{2(k(n-1)+1)} & \text{if } k \text{ is even or } n \text{ is odd,} \\ \dfrac{1}{2(k(n-1)+2)} & \text{otherwise.} \end{cases} \tag{7.1}$$

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